

# The Picard Group of Corings

Mohssin Zarouali-Darkaoui

## Introduction

Corings were introduced by Sweedler in [25]. A coring over an associative algebra with unit over a commutative ring  $k$ ,  $A$ , is an  $A$ -bimodule  $\mathfrak{C}$  with two  $A$ -bimodule maps  $\Delta_{\mathfrak{C}} : \mathfrak{C} \rightarrow \mathfrak{C} \otimes_A \mathfrak{C}$  (comultiplication) and  $\epsilon_{\mathfrak{C}} : \mathfrak{C} \rightarrow A$  (counit) such that the same diagrams as for coalgebras are commutative. Recently, corings were intensively studied. The main motivation of this studies is the observation of Takeuchi, Theorem 4.1. For a detailed study of corings, we refer to [6].

The connection between the Picard group of Azumaya algebras and its automorphisms were studied in [22]. Bass generalized these connections for arbitrary algebras, see [2]. Further study of the Picard group of algebras is given in [13]. In [3], the authors have given the versions of these connections for rings with local units, in order to study the Picard group of the category  $R - gr$ , where  $R$  is a  $G$ -graded ring. In [23], the authors studied these connections for coalgebras over fields.

The purpose of this note is to introduce and study the right Picard group of corings. The motivation is the fact that there is an isomorphism of groups between the Picard group of the category  $\mathcal{M}^{\mathfrak{C}}$ , for a certain coring  $\mathfrak{C}$ , and the right Picard group of  $\mathfrak{C}$  which is defined as the group of the isomorphism classes of right invertible  $\mathfrak{C}$ -bicomodules ( $=T(\mathfrak{C})$  for some  $k$ -autoequivalence of  $\mathcal{M}^{\mathfrak{C}}$ ,  $T$ ) with the composition law induced by the cotensor product (see Proposition 2.3). We extend the exact sequence  $1 \rightarrow \text{Inn}_k(-) \rightarrow \text{Aut}_k(-) \rightarrow \text{Pic}_k(-)$  from algebras and coalgebras over fields to corings. We also extend a result which is useful to show that a given coring  $\mathfrak{C}$  have the right Aut-Pic property, i.e. where the morphism  $\text{Aut}_k(\mathfrak{C}) \rightarrow \text{Pic}_k^r(\mathfrak{C})$  is an epimorphism (see Proposition 2.7). Of course, our Aut-Pic property for corings extends that of algebras [4], and that of coalgebras over fields [10]. All of the examples of algebras and coalgebras having the Aut-Pic property that are given in [4, 10], are corings having this property. In this note we give some new examples of corings having the Aut-Pic property. We also simplify the computation of the right Picard group of several interesting corings (see Proposition 3.6). Finally, in section 4, we give the corresponding exact sequences for the category of entwined modules over an entwining structure, the category of Doi-Koppinen-Hopf modules over a Doi-Koppinen structure, and the category of graded modules by a  $G$ -set, where  $G$  is a group.

# 1 Preliminaries

Throughout this note and unless otherwise stated,  $k$  denote a commutative ring (with unit),  $A, A', A'', A_1, A_2, B, B_1$ , and  $B_2$  denote associative and unitary algebras over  $k$ , and  $\mathfrak{C}, \mathfrak{C}', \mathfrak{C}'', \mathfrak{C}_1, \mathfrak{C}_2, \mathfrak{D}, \mathfrak{D}_1$  and  $\mathfrak{D}_2$  denote corings over  $A, A', A'', A_1, A_2, B, B_1$ , and  $B_2$ , respectively. The notation  $\otimes$  will stand for the tensor product over  $k$ .

A category  $\mathbf{C}$  is said to be  $k$ -category if for every  $M$  and  $N$  in  $\mathbf{C}$ ,  $\text{Hom}_{\mathbf{C}}(M, N)$  is a  $k$ -module, and the composition is  $k$ -bilinear. An abelian category which is a  $k$ -category is said to be  $k$ -abelian category. A functor between  $k$ -categories is said to be  $k$ -functor or  $k$ -linear functor if it is  $k$ -linear on the  $k$ -modules of morphisms. A functor between  $k$ -categories is said to be a  $k$ -equivalence if it is  $k$ -linear and an equivalence.

We recall from [25] that an  $A$ -coring is an  $A$ -bimodule  $\mathfrak{C}$  with two  $A$ -bimodule maps  $\Delta : \mathfrak{C} \rightarrow \mathfrak{C} \otimes_A \mathfrak{C}$  and  $\epsilon : \mathfrak{C} \rightarrow A$  such that  $(\mathfrak{C} \otimes_A \Delta) \circ \Delta = (\Delta \otimes_A \mathfrak{C}) \circ \Delta$  and  $(\epsilon \otimes_A \mathfrak{C}) \circ \Delta = (\mathfrak{C} \otimes_A \epsilon) \circ \Delta = 1_{\mathfrak{C}}$ .  $\mathfrak{C} = A$  endowed with the obvious structure maps is an  $A$ -coring. A right  $\mathfrak{C}$ -comodule is a pair  $(M, \rho_M)$  where  $M$  is a right  $A$ -module and  $\rho_M : M \rightarrow M \otimes_A \mathfrak{C}$  (coaction) is an  $A$ -linear map satisfying  $(M \otimes_A \Delta) \circ \rho_M = (\rho_M \otimes_A \mathfrak{C}) \circ \rho_M$ , and  $(M \otimes_A \epsilon) \circ \rho_M = 1_M$ . A morphism of right  $\mathfrak{C}$ -comodules  $(M, \rho_M)$  and  $(N, \rho_N)$  is a right  $A$ -linear map  $f : M \rightarrow N$  such that  $(f \otimes_A \mathfrak{C}) \circ \rho_M = \rho_N \circ f$ . Right  $\mathfrak{C}$ -comodules with their morphisms form a  $k$ -category  $\mathcal{M}^{\mathfrak{C}}$ . Coproducts and cokernels (and then inductive limits) in  $\mathcal{M}^{\mathfrak{C}}$  exist and they coincide respectively with coproducts and cokernels in the category of right  $A$ -modules  $\mathcal{M}_A$ . If  ${}_A \mathfrak{C}$  is flat, then  $\mathcal{M}^{\mathfrak{C}}$  is a  $k$ -abelian category. Moreover it is a Grothendieck category. When  $\mathfrak{C} = A$  is the obvious  $A$ -coring,  $\mathcal{M}^A$  is the category of right  $A$ -modules  $\mathcal{M}_A$ .

Now assume that the  $A' - A$ -bimodule  $M$  is also a left comodule over an  $A'$ -coring  $\mathfrak{C}'$  with structure map  $\lambda_M : M \rightarrow \mathfrak{C}' \otimes_{A'} M$ . Assume moreover that  $\rho_M$  is  $A'$ -linear, and  $\lambda_M$  is  $A$ -linear. It is clear that  $\rho_M : M \rightarrow M \otimes_A \mathfrak{C}$  is a morphism of left  $\mathfrak{C}'$ -comodules if and only if  $\lambda_M : M \rightarrow \mathfrak{C}' \otimes_{A'} M$  is a morphism of right  $\mathfrak{C}$ -comodules. In this case, we say that  $M$  is a  $\mathfrak{C}' - \mathfrak{C}$ -bicomodule. A morphism of bicomodules is a morphism of right and left comodules. Then we obtain a  $k$ -category  ${}^{\mathfrak{C}'} \mathcal{M}^{\mathfrak{C}}$ . If  ${}_A \mathfrak{C}$  and  $\mathfrak{C}'_{A'}$  it is a Grothendieck category. If  $\mathfrak{C}' = A', \mathfrak{C} = A$ , then  ${}^{\mathfrak{C}'} \mathcal{M}^{\mathfrak{C}}$  is the category of  $A' - A$ -bimodules,  ${}_{A'} \mathcal{M}_A$ .

A coring  $\mathfrak{C}$  is said to be coseparable if the comultiplication map  $\Delta_{\mathfrak{C}}$  is a section in the category  ${}^{\mathfrak{C}} \mathcal{M}^{\mathfrak{C}}$ . Obviously the trivial  $A$ -coring  $\mathfrak{C} = A$  is coseparable.

Let  $Z$  be a left  $A$ -module and  $f : X \rightarrow Y$  a morphism in  $\mathcal{M}_A$ . Following [6, 40.13] we say that  $f$  is  $Z$ -pure when the functor  $- \otimes_A Z$  preserves the kernel of  $f$ . If  $f$  is  $Z$ -pure for every  $Z \in {}_A \mathcal{M}$  then we say simply that  $f$  is pure in  $\mathcal{M}_A$ .

Let  $f : M \rightarrow N$  be a morphism in  ${}^{\mathfrak{C}'} \mathcal{M}^{\mathfrak{C}}$ , and let  $\ker(f)$  be its kernel in  ${}_{A'} \mathcal{M}_A$ . It is easy to show that if  $\ker(f)$  is  $\mathfrak{C}'_{A'}$ -pure and  ${}_A \mathfrak{C}$ -pure, and the following

$$\ker(f) \otimes_A \mathfrak{C} \otimes_A \mathfrak{C}, \quad \mathfrak{C}' \otimes_{A'} \mathfrak{C}' \otimes_{A'} \ker(f) \quad \text{and} \quad \mathfrak{C}' \otimes_{A'} \ker(f) \otimes_A \mathfrak{C}$$

are injective maps, then  $\ker(f)$  is the kernel of  $f$  in  ${}^{\mathfrak{C}'} \mathcal{M}^{\mathfrak{C}}$ . This is the case if  $f$  is  $(\mathfrak{C}' \otimes_{A'} \mathfrak{C}')_{A'}$ -pure,  ${}_A (\mathfrak{C} \otimes_A \mathfrak{C})$ -pure, and  $\mathfrak{C}' \otimes_{A'} f$  is  ${}_A \mathfrak{C}$ -pure (e.g. if  $\mathfrak{C}'_{A'}$  and  ${}_A \mathfrak{C}$  are flat, or if  $\mathfrak{C}$  is a coseparable  $A$ -coring).

Now let  $M \in {}^{\mathfrak{C}'}\mathcal{M}^{\mathfrak{C}}$  and  $N \in {}^{\mathfrak{C}}\mathcal{M}^{\mathfrak{C}''}$ . The map

$$\omega_{M,N} = \rho_M \otimes_A N - M \otimes_A \lambda_N : M \otimes_A N \rightarrow M \otimes_A \mathfrak{C} \otimes_A N$$

is a  $\mathfrak{C}' - \mathfrak{C}''$ -bicomodule map. Its kernel in  ${}_A\mathcal{M}_{A''}$  is the *cotensor product* of  $M$  and  $N$ , and it is denoted by  $M \square_{\mathfrak{C}} N$ . From the above consideration, if  $\omega_{M,N}$  is  $(\mathfrak{C}' \otimes_{A'} \mathfrak{C}')_{A'}$ -pure,  ${}_{A''}(\mathfrak{C}'' \otimes_{A''} \mathfrak{C}'')$ -pure, and  $\mathfrak{C}' \otimes_{A'} \omega_{M,N}$  is  ${}_{A''}\mathfrak{C}''$ -pure (e.g. if  $\mathfrak{C}'_{A'}$  and  ${}_{A''}\mathfrak{C}''$  are flat, or if  $\mathfrak{C}$  is a coseparable  $A$ -coring), then  $M \square_{\mathfrak{C}} N$  is the kernel of  $\omega_{M,N}$  in  ${}^{\mathfrak{C}'}\mathcal{M}^{\mathfrak{C}''}$ .

If for every  $M \in {}^{\mathfrak{C}'}\mathcal{M}^{\mathfrak{C}}$  and  $N \in {}^{\mathfrak{C}}\mathcal{M}^{\mathfrak{C}''}$ ,  $\omega_{M,N}$  is  $\mathfrak{C}'_{A'}$ -pure and  ${}_{A''}\mathfrak{C}''$ -pure, then we have a  $k$ -linear bifunctor

$$-\square_{\mathfrak{C}}- : {}^{\mathfrak{C}'}\mathcal{M}^{\mathfrak{C}} \times {}^{\mathfrak{C}}\mathcal{M}^{\mathfrak{C}''} \longrightarrow {}^{\mathfrak{C}'}\mathcal{M}^{\mathfrak{C}''}. \quad (1)$$

If in particular  $\mathfrak{C}'_{A'}$  and  ${}_{A''}\mathfrak{C}''$  are flat, or if  $\mathfrak{C}$  is a coseparable  $A$ -coring, then the bifunctor (1) is well defined. In the special case when  $\mathfrak{C} = A$ ,  $-\square_{\mathfrak{C}}- = - \otimes_A -$ .

Throughout this note, for all coring  $\mathfrak{C}$  over  $A$ , we have  ${}_A\mathfrak{C}$  and  $\mathfrak{C}_A$  are flat.

## 2 The Picard group of corings

A *coring homomorphism* [14] from the coring  $(\mathfrak{C} : A)$  to the coring  $(\mathfrak{D} : B)$  is a pair  $f = (\varphi, \rho)$ , where  $\rho : A \rightarrow B$  is a homomorphism of  $k$ -algebras and  $\varphi : \mathfrak{C} \rightarrow \mathfrak{D}$  is a homomorphism of  $A$ -bimodules such that

$$\epsilon_{\mathfrak{D}} \circ \varphi = \rho \circ \epsilon_{\mathfrak{C}} \quad \text{and} \quad \Delta_{\mathfrak{D}} \circ \varphi = \omega_{\mathfrak{D},\mathfrak{D}} \circ (\varphi \otimes_A \varphi) \circ \Delta_{\mathfrak{C}},$$

where  $\omega_{\mathfrak{D},\mathfrak{D}} : \mathfrak{D} \otimes_A \mathfrak{D} \rightarrow \mathfrak{D} \otimes_B \mathfrak{D}$  is the canonical map induced by  $\rho : A \rightarrow B$ . Corings over  $k$ -algebras and their morphisms form a category. We denote it by  $\mathbf{Crg}_k$ . Notice that the category of  $k$ -algebras,  $\mathbf{Alg}_k$ , and the category of  $k$ -coalgebras,  $\mathbf{Coalg}_k$ , are full subcategories of  $\mathbf{Crg}_k$ . Notice also that  $(\varphi, \rho) \in \mathbf{Crg}_k$  is an isomorphism if and only if  $\varphi$  and  $\rho$  are bijective.

To state the definition of the right Picard group of a coring we need to recall the following result.

**Proposition 2.1.** ([27, Proposition 3.1]) *Suppose that  ${}_A\mathfrak{C}$ ,  $\mathfrak{C}_A$ ,  ${}_B\mathfrak{D}$  and  $\mathfrak{D}_B$  are flat. Let  $X \in {}^{\mathfrak{C}}\mathcal{M}^{\mathfrak{D}}$  and  $\Lambda \in {}^{\mathfrak{D}}\mathcal{M}^{\mathfrak{C}}$ .*

*The following statements are equivalent:*

- (1)  $(-\square_{\mathfrak{C}}X, -\square_{\mathfrak{D}}\Lambda)$  is a pair of inverse equivalences;
- (2) there exist bicomodule isomorphisms

$$f : X \square_{\mathfrak{D}} \Lambda \rightarrow \mathfrak{C} \quad \text{and} \quad g : \Lambda \square_{\mathfrak{C}} X \rightarrow \mathfrak{D}$$

*in  ${}^{\mathfrak{C}}\mathcal{M}^{\mathfrak{C}}$  and  ${}^{\mathfrak{D}}\mathcal{M}^{\mathfrak{D}}$  respectively, such that*

- (a)  ${}_A X$  and  ${}_B \Lambda$  are flat, and  $\omega_{X,\Lambda} = \rho_X \otimes_B \Lambda - X \otimes_A \rho_\Lambda$  is pure in  ${}_A \mathcal{M}$  and  $\omega_{\Lambda,X} = \rho_\Lambda \otimes_A X - \Lambda \otimes_B \rho_X$  is pure in  ${}_B \mathcal{M}$ , or
- (b)  $\mathfrak{C}X$  and  $\mathfrak{D}\Lambda$  are coflat.

In such a case the diagrams

$$\begin{array}{ccc}
\Lambda \square_{\mathfrak{C}} X \square_{\mathfrak{D}} \Lambda & \xrightarrow{\Lambda \square_{\mathfrak{C}} f} & \Lambda \square_{\mathfrak{C}} \mathfrak{C} \\
g \square_{\mathfrak{D}} \Lambda \downarrow & & \downarrow \simeq \\
\mathfrak{D} \square_{\mathfrak{D}} \Lambda & \xrightarrow{\simeq} & \Lambda
\end{array}
\quad
\begin{array}{ccc}
X \square_{\mathfrak{D}} \Lambda \square_{\mathfrak{C}} X & \xrightarrow{f \square_{\mathfrak{C}} X} & \mathfrak{C} \square_{\mathfrak{C}} X \\
X \square_{\mathfrak{D}} g \downarrow & & \downarrow \simeq \\
X \square_{\mathfrak{D}} \mathfrak{D} & \xrightarrow{\simeq} & X
\end{array}$$

commute.

If  $A$  and  $B$  are von Neumann regular rings, or if  $\mathfrak{C}$  and  $\mathfrak{D}$  are coseparable corings (without the condition “ ${}_A \mathfrak{C}$ ,  $\mathfrak{C}_A$ ,  ${}_B \mathfrak{D}$  and  $\mathfrak{D}_B$  are flat”), the conditions (a) and (b) can be deleted.

Let  $\mathbf{C}$  be a  $k$ -abelian category,  $\text{Pic}_k(\mathbf{C})$  [2] is the group of isomorphism classes  $(T)$  of  $k$ -equivalences  $T : \mathbf{C} \rightarrow \mathbf{C}$  with the composition law  $(T)(S) = (ST)$ .

**Definition 2.2.** We say that a  $\mathfrak{C} - \mathfrak{D}$ -bicomodule  $X$  is *right invertible* if the functor  $-\square_{\mathfrak{C}} X : \mathcal{M}^{\mathfrak{C}} \rightarrow \mathcal{M}^{\mathfrak{D}}$  is an equivalence. We say that  $\mathfrak{C}$  and  $\mathfrak{D}$  are *Morita-Takeuchi right equivalent* if there is a right invertible  $\mathfrak{C} - \mathfrak{D}$ -bicomodule.

The isomorphism classes  $(X)$  of right invertible  $\mathfrak{C}$ -bicomodules with the composition law

$$(X_1)(X_2) = (X_1 \square_{\mathfrak{C}} X_2),$$

form the *right Picard group* of  $\mathfrak{C}$ . We denote it by  $\text{Pic}_k^r(\mathfrak{C})$ . This law is well defined since  $-\square_{\mathfrak{C}}(X_1 \square_{\mathfrak{C}} X_2) \simeq (-\square_{\mathfrak{C}} X_1) \square_{\mathfrak{C}} X_2$  (From [6, 22.5, 22.6(iii)] and  $\mathfrak{C}X_2$  is coflat). The associativity of this law follows from the associativity of the cotensor product ( $\mathfrak{C}X_1$  and  $\mathfrak{C}X_2$  are coflat). The isomorphism class  $(\mathfrak{C})$  is the identity element.  $(X)^{-1} = (\Lambda)$ , where  $\Lambda$  is such that  $(-\square_{\mathfrak{C}} X, -\square_{\mathfrak{C}} \Lambda)$  is a pair of inverse equivalences.

It follows from [27, Theorems 3.8, 3.13] that if  $\mathfrak{C}$  is coseparable (this case includes that of algebras), or cosemisimple, or  $A$  is von Neumann regular ring, or if  $\mathfrak{C}$  is a coalgebra over a QF ring, then a  $\mathfrak{C}$ -bicomodule  $X$  is right invertible if and only if it is left invertible. Hence, for each case we have  $\text{Pic}_k^r(\mathfrak{C}) = \text{Pic}_k^l(\mathfrak{C})$ . We will denote this last by  $\text{Pic}_k(\mathfrak{C})$ .

The following proposition is the motivation of the study of the Picard group of corings. The proof is straightforward using [26, Theorem 2.3].

**Proposition 2.3.** *There are inverse isomorphisms of groups*

$$\text{Pic}_k^r(\mathfrak{C}) \xrightleftharpoons[\beta]{\alpha} \text{Pic}_k(\mathcal{M}^{\mathfrak{C}}),$$

$$\alpha((X)) = (-\square_{\mathfrak{C}} X) \text{ and } \beta((T)) = (T(\mathfrak{C})).$$

Now we introduce two categories,  $\mathbf{Crg}_k$  and  $\mathcal{MT} - \mathbf{Crg}_k^r$ . The objects of both are the corings. The morphisms of  $\mathbf{Crg}_k$  are the isomorphisms of corings. The morphism of  $\mathcal{MT} - \mathbf{Crg}_k^r$  are the Morita-Takeuchi right equivalences. A *right Morita-Takeuchi equivalence*  $\mathfrak{C} \sim \mathfrak{D}$  is simply an isomorphism class  $(M)$  where  ${}_M \mathfrak{C}$  is a right invertible  $\mathfrak{D} - \mathfrak{C}$ -bicomodule. If  $(N)$  is a Morita-Takeuchi right equivalence  $\mathfrak{D} \sim \mathfrak{C}'$ , the composite is given by  $(N \square_{\mathfrak{D}} M)$ .

The categories  $\mathbf{Crg}_k$  and  $\mathcal{MT} - \mathbf{Crg}_k^r$  are groupoids (i.e., categories in which every morphism is an isomorphism). Then  $\text{Aut}_k(\mathfrak{C})$  and  $\text{Pic}_k^r(\mathfrak{C})$  are respectively the endomorphism group of the object  $\mathfrak{C}$  in  $\mathbf{Crg}_k$  and in  $\mathcal{MT} - \mathbf{Crg}_k^r$ .

Let  $f : (\mathfrak{C} : A) \rightarrow (\mathfrak{D} : B)$  and  $g : (\mathfrak{C}_1 : A_1) \rightarrow (\mathfrak{D}_1 : B_1)$  be two morphisms of corings, and let  ${}_A X_{\mathfrak{C}}$  be a bicomodule.

We consider the right induction functor  $- \otimes_A B : \mathcal{M}^{\mathfrak{C}} \rightarrow \mathcal{M}^{\mathfrak{D}}$  defined in [14]. The right coaction on the right  $B$ -module  $M \otimes_A B$  is given by:

$$\rho_{M \otimes_A B} : M \otimes_A B \rightarrow M \otimes_A B \otimes_B \mathfrak{D} \simeq M \otimes_A \mathfrak{D}, \quad m \otimes_A b \mapsto \sum m_{(0)} \otimes_A \varphi(m_{(1)})b,$$

where  $\rho_M(m) = \sum m_{(0)} \otimes_A m_{(1)}$ . By a similar way we define the left induction functor  $B_1 \otimes_{A_1} - : {}_{\mathfrak{C}_1} \mathcal{M} \rightarrow {}_{\mathfrak{D}_1} \mathcal{M}$ .

Now consider  ${}_g X_f := B_1 \otimes_{A_1} X \otimes_A B$  which is a  $\mathfrak{D}_1 - \mathfrak{D}$ -bicomodule, and  $X_f := X \otimes_A B$  which is a  $\mathfrak{C}_1 - \mathfrak{D}$ -bicomodule. It can be showed easily that  ${}_1 X_f \simeq X_f$  as bicomodules, and that if  $\mathfrak{C} \xrightarrow{f=(\varphi, \rho)} \mathfrak{D} \xrightarrow{f'=(\varphi', \rho')} \mathfrak{C}'$  are morphisms of corings, then  $(X_f)_{f'} \simeq X_{f'f}$  as bicomodules. Moreover, from [26, Theorem 2.4],  $X_f \simeq X \square_{\mathfrak{C}} \mathfrak{C}_f$  as  $\mathfrak{C}_1 - \mathfrak{D}$ -bicomodules. Finally, it is easy to show that if  $f$  is an isomorphism, then  $X_f \simeq (X_f)'$  as bicomodules, where  $(X_f)' = X_f$ , and the left  $\mathfrak{C}_1$ -comodule structure on  $(X_f)'$  is the same as for  $X_f$ . The right  $B$ -module structure on  $(X_f)'$  is given by  $xb = x\rho^{-1}(b)$ , for  $x \in X, b \in B$ . The right  $\mathfrak{D}$ -comodule structure on it is given by  $\rho_{(X_f)'}(x) = \sum x_{(0)} \otimes_B \varphi(x_{(1)})$ , where  $x \in X$  and  $\rho_X(x) = \sum x_{(0)} \otimes_A x_{(1)}$ .

**Lemma 2.4.** *We have a functor*

$$\Omega : \mathbf{Crg}_k \rightarrow \mathcal{MT} - \mathbf{Crg}_k^r,$$

*which is the identity on objects and associates with an isomorphism of corings  $f : (\mathfrak{C} : A) \rightarrow (\mathfrak{D} : B)$  the isomorphism class of the right invertible  $\mathfrak{D} - \mathfrak{C}$ -bicomodule  ${}_f \mathfrak{C}$ .*

*Proof.* Let  $f = (\varphi, \rho) : (\mathfrak{C} : A) \rightarrow (\mathfrak{D} : B)$  be an isomorphism of corings. Since  $B \otimes_A \mathfrak{C} \otimes_A B \rightarrow \mathfrak{D}$ ,  $b \otimes_A c \otimes_A b \mapsto b\varphi(c)b$  is a morphism of  $B$ -corings (see [6, 24.1]), it follows from [27, Theorem 4.1] that  ${}_f \mathfrak{C}$  is a right invertible  $\mathfrak{D} - \mathfrak{C}$ -bicomodule. Now let  $\mathfrak{C} \xrightarrow{f} \mathfrak{D} \xrightarrow{g} \mathfrak{C}'$  be two morphisms in  $\mathbf{Crg}_k$ . Then  ${}_g f \mathfrak{C} \simeq {}_g({}_f \mathfrak{C}) \simeq {}_g \mathfrak{D} \square_{\mathfrak{D}} {}_f \mathfrak{C}$  is an isomorphism of  $\mathfrak{C}' - \mathfrak{C}$ -bicomodules.  $\square$

Now we are ready to state and prove the main result of this section.

**Theorem 2.5.** *Let  $\mathfrak{C}$  be an  $A$ -coring such that  ${}_A \mathfrak{C}$  and  $\mathfrak{C}_A$  are flat.*

(1) We have an exact sequence

$$1 \longrightarrow \text{Inn}_k^r(\mathfrak{C}) \longrightarrow \text{Aut}_k(\mathfrak{C}) \xrightarrow{\Omega} \text{Pic}_k^r(\mathfrak{C}),$$

where  $\text{Inn}_k^r(\mathfrak{C})$  is the set of  $f = (\varphi, \rho) \in \text{Aut}_k(\mathfrak{C})$  such that there is  $p \in \mathfrak{C}^*$  invertible with

$$\sum \varphi(c_{(1)})p(c_{(2)}) = \sum p(c_{(1)})c_{(2)},$$

for every  $c \in \mathfrak{C}$ . As for algebras and coalgebras over fields, we call  $\text{Inn}_k^r(\mathfrak{C})$  the right group of inner automorphisms of  $\mathfrak{C}$

In particular there is a monomorphism from the quotient group  $\text{Out}_k^r(\mathfrak{C}) := \text{Aut}_k(\mathfrak{C})/\text{Inn}_k^r(\mathfrak{C})$  to  $\text{Pic}_k^r(\mathfrak{C})$ . As for algebras and coalgebras over fields, we call  $\text{Out}_k^r(\mathfrak{C})$  the right outer automorphism group of  $\mathfrak{C}$ .

(2) If  $\mathfrak{C}$  and  $\mathfrak{D}$  are Morita-Takeuchi right equivalent, then  $\text{Pic}_k^r(\mathfrak{C}) \simeq \text{Pic}_k^r(\mathfrak{D})$ .

*Proof.* (1) Let  $f = (\varphi, \rho) \in \text{Aut}_k(\mathfrak{C})$  and  ${}_f\mathfrak{C} \simeq \mathfrak{C}$  as  $\mathfrak{C}$ -bicomodules. We have  ${}_f\mathfrak{C} \simeq ({}_f\mathfrak{C})'$  as  $\mathfrak{C}$ -bicomodules (see the remark just before Lemma 2.4). Now let  $h : ({}_f\mathfrak{C})' \rightarrow \mathfrak{C}$  be an isomorphism of  $\mathfrak{C}$ -bicomodules.  $h$  is a morphism of right  $\mathfrak{C}$ -bicomodules if and only if there is  $p \in \mathfrak{C}^*$  invertible such that  $h(c) = \sum p(c_{(1)})c_{(2)}$ , for every  $c \in \mathfrak{C}$  (see [6, 18.12(1)]). In such a case,  $p = \epsilon h$ .  $h$  is a morphism of left  $\mathfrak{C}$ -bicomodules means that

$$h(ac) = \rho(a)h(c), \quad (*)$$

for every  $a \in A, c \in \mathfrak{C}$ , and for every  $c \in \mathfrak{C}$ ,

$$\sum \varphi(c_{(1)}) \otimes_A h(c_{(2)}) = \sum h(c)_{(1)} \otimes_A h(c)_{(2)}. \quad (**)$$

The condition (\*\*) is equivalent to

$$\sum \varphi(c_{(1)}) \otimes_A p(c_{(2)})c_{(3)} = \sum p(c_{(1)})c_{(2)} \otimes_A c_{(3)},$$

for every  $c \in \mathfrak{C}$ . Hence

$$\sum \varphi(c_{(1)})p(c_{(2)}) = \sum p(c_{(1)})c_{(2)},$$

for every  $c \in \mathfrak{C}$ . Since  $\varphi$  is left  $A$ -linear, (\*) holds. The converse is obvious.

(2) It follows immediately from that  $\mathfrak{C}$  and  $\mathfrak{D}$  are isomorphic to each other in the category  $\mathfrak{MT} - \mathfrak{Crg}_k^r$ . More explicitly, Let  ${}_c\mathcal{M}_{\mathfrak{D}}$  be a right invertible bicomodule. The map

$$(X) \mapsto (M^{-1} \square_{\mathfrak{C}} X \square_{\mathfrak{C}} M)$$

is an isomorphism from  $\text{Pic}_k^r(\mathfrak{C})$  to  $\text{Pic}_k^r(\mathfrak{D})$ . □

The last theorem gives a well-know result of Bass (see [2, Proposition II (5.2)(3)] or [11, Theorems 55.9, 55.11]), and a generalization to the case of coalgebras over rings of the particular case of [23, Theorem 2.7] where  $R = k$ .

**Corollary 2.6.** (1) For a  $k$ -algebra  $A$ , we have an exact sequence

$$1 \longrightarrow \text{Inn}_k(A) \longrightarrow \text{Aut}_k(A) \xrightarrow{\Omega} \text{Pic}_k(A),$$

where  $\text{Inn}_k(A) := \{a \mapsto bab^{-1} \mid b \text{ invertible in } A\}$ , the group of inner automorphisms of  $A$ .

(2) For a  $k$ -coalgebra  $C$  such that  ${}_k C$  is flat, we have an exact sequence

$$1 \longrightarrow \text{Inn}_k(C) \longrightarrow \text{Aut}_k(C) \xrightarrow{\Omega} \text{Pic}_k^r(C),$$

where  $\text{Inn}_k(C)$  (the group of inner automorphisms of  $C$ ) is the set of  $\varphi \in \text{Aut}_k(C)$  such that there is  $p \in C^*$  invertible with  $\varphi(c) = \sum p(c_{(1)})c_{(2)}p^{-1}(c_{(3)})$ , for every  $c \in C$ .

*Proof.* (1) It suffices to see that for  $p \in \mathfrak{C}^*$  where  $\mathfrak{C} = A$ ,  $p$  is invertible if and only if  $p = \lambda_b : a \mapsto ba$  and  $b$  is invertible in  $A$ .

(2) Suppose that  $\varphi \in \text{Aut}_k(C)$  and  $\sum \varphi(c_{(1)})p(c_{(2)}) = \sum p(c_{(1)})c_{(2)}$ , for every  $c \in C$ . Then  $\varphi(c) = \sum \varphi(c_{(1)})\epsilon_C(c_{(2)}) = \sum \varphi(c_{(1)})p(c_{(2)})p^{-1}(c_{(3)}) = \sum p(c_{(1)})c_{(2)}p^{-1}(c_{(3)})$ , for every  $c \in C$ . The converse is obvious.  $\square$

In order to consider the one-sided comodule structure of invertible bicomodules we need

**Proposition 2.7.** Let  $(M), (N) \in \text{Pic}_k^r(\mathfrak{C})$ . Then

(1)  $(N) \in \text{Im}\Omega.(M)$  if and only if  $N \simeq_f M$  as bicomodules for some  $f \in \text{Aut}_k(\mathfrak{C})$ .

(2)  ${}_A M_{\mathfrak{C}} \simeq {}_A N_{\mathfrak{C}}$  if and only if  $N \simeq_f M$  as bicomodules for some  $f = (\varphi, 1_A) \in \text{Aut}_k(\mathfrak{C})$ .

*Proof.* (1) Straightforward from the fact that for every  $f \in \text{Aut}_k(\mathfrak{C})$ , there is an isomorphism of  $\mathfrak{C}$ -bicomodules  ${}_f M \simeq {}_f \mathfrak{C} \square_{\mathfrak{C}} M$ .

(2) ( $\Leftarrow$ ) Obvious. ( $\Rightarrow$ ) let  $h : {}_A M_{\mathfrak{C}} \rightarrow {}_A N_{\mathfrak{C}}$  be a bicomodule isomorphism. Since  $M$  is quasi-finite as a right  $\mathfrak{C}$ -comodule (see [27, Proposition 3.6]), it has a structure of  $e_{\mathfrak{C}}(M) - \mathfrak{C}$ -bicomodule. Then  $N$  has a structure of  $e_{\mathfrak{C}}(M) - \mathfrak{C}$ -bicomodule induced by  $h$ , and  $h$  is an  $e_{\mathfrak{C}}(M) - \mathfrak{C}$ -bicomodule isomorphism. On the other hand,  $N$  is quasi-finite as a right  $\mathfrak{C}$ -comodule. Let  $F =: h_{\mathfrak{C}}(N, -) : \mathcal{M}^{\mathfrak{C}} \rightarrow \mathcal{M}^{e_{\mathfrak{C}}(M)}$  be the cohom functor, and let  $\theta : 1_{\mathcal{M}^{\mathfrak{C}}} \rightarrow F(-) \square_{e_{\mathfrak{C}}(M)} N$  and  $\chi : F(- \square_{e_{\mathfrak{C}}(M)} N) \rightarrow 1_{\mathcal{M}^{e_{\mathfrak{C}}(M)}}$  be respectively the unit and the counit of this adjunction. From [12, Proposition 5.2] and its proof, the map  $\varphi := \chi_{e_{\mathfrak{C}}(M)} \circ F(\lambda_N^h) : e_{\mathfrak{C}}(N) \rightarrow e_{\mathfrak{C}}(M)$ , where  $\lambda_N^h : N \rightarrow e_{\mathfrak{C}}(M) \square_{e_{\mathfrak{C}}(M)} N$  is the left comodule structure map on  $N$ , is a morphism of  $A$ -corings, and making commutative the diagram

$$\begin{array}{ccc} N & \xrightarrow{\theta_N} & e_{\mathfrak{C}}(N) \otimes_A N \\ & \searrow \lambda_N^h & \downarrow \varphi \otimes_A 1_N \\ & & e_{\mathfrak{C}}(M) \otimes_A N. \end{array}$$

Since  $F$  is a left adjoint to  $-\square_{e_{\mathfrak{C}}(M)}N$  which is an equivalence,  $\chi_{e_{\mathfrak{C}}(M)}$  and  $\varphi$  are isomorphisms. Since  $M$  and  $N$  are right invertibles,  $e_{\mathfrak{C}}(M) \simeq \mathfrak{C} \simeq e_{\mathfrak{C}}(N)$  (see [27, Proposition 3.6]). We identify  $e_{\mathfrak{C}}(M)$  and  $e_{\mathfrak{C}}(N)$  with  $\mathfrak{C}$ . Then  $\varphi$  is a coring endomorphism of  $\mathfrak{C}$ . Hence  $M \simeq_f N$  as  $\mathfrak{C}$ -bicomodules, where  $f = (\varphi, 1_A)$ .  $\square$

As a immediate consequence of the last result, we obtain a generalization to the case of coalgebras over rings of the particular case of [23, Proposition 2.8] where  $R = k$ . Its version for algebras is due to Bass, see [2, Proposition II (5.2)(4)] or [11, Theorem 55.12].

**Corollary 2.8.** *Let  $C$  be a  $k$ -coalgebra such that  ${}_k C$  is flat, and let  $(M), (N) \in \text{Pic}_k^r(C)$ . Then  $M_C \simeq N_C$  if and only if  $(N) \in \text{Im} \Omega.(M)$ , that is,  $N \simeq_f M$  as bicomodules for some  $f \in \text{Aut}_k(C)$ .*

### 3 The Aut-Pic Property

Following [4] and [10], we give the following

**Definition 3.1.** We say that a coring  $\mathfrak{C}$  has the *right Aut-Pic property* if the group morphism  $\Omega$  of Theorem 2.5 is surjective. In such a case,  $\text{Out}_k^r(\mathfrak{C}) \simeq \text{Pic}_k^r(\mathfrak{C})$ .

Bolla proved ([4, p. 264]) that every ring such that all right progenerators (=finitely generated projective generators) are free, has Aut-Pic. In particular, local rings (by using a theorem of Kaplansky), principal right ideal domains (by [15, Corollary 2.27]), and polynomial rings  $k[X_1, \dots, X_n]$ , where  $k$  is a field (by Quillen-Suslin Theorem), have Aut-Pic. He also proved that every basic (semiperfect) ring has Aut-Pic (see [4, Proposition 3.8]). Moreover, it is well-known that a semiperfect ring  $A$  is Morita equivalent to its basic ring  $eAe$  (see [1, Proposition 27.14]), and by Theorem 2.5,  $\text{Pic}_k(A) \simeq \text{Out}_k(eAe)$ . In [10], the authors gave several interesting examples of coalgebras over fields having Aut-Pic. For instance, they proved that every basic coalgebra has Aut-Pic. On the other hand, we know [9, Corollary 2.2] that given a coalgebra over a field  $C$  is Morita-Takeuchi equivalent to a basic coalgebra  $C_0$ , and by Theorem 2.5,  $\text{Pic}_k(C) \simeq \text{Out}_k(C_0)$ .

Of course all of the examples of rings and coalgebras over fields having Aut-Pic mentioned in [4] and [10] are examples of corings having Aut-Pic. In order to give others examples of corings satisfying the Aut-Pic property we need the next lemma which is a generalization of [18, Proposition 4.1].

We recall from [18], that an object  $M$  in an additive category  $\mathbf{C}$  has the *invariant basis number property* (IBN for short) if  $M^n \simeq M^m$  implies  $n = m$ . For example every non-zero finitely generated projective module over a semiperfect ring has IBN (from [1, Theorem 27.11]). A bicomodule  $M \in {}_B \mathcal{M}^{\mathfrak{C}}$  is said to be  $(B, \mathfrak{C})$ -*quasi-finite* [6] if the functor  $-\otimes_B M : \mathcal{M}_B \rightarrow \mathcal{M}^{\mathfrak{C}}$  has a left adjoint  $\text{h}_{\mathfrak{C}}(M, -) : \mathcal{M}^{\mathfrak{C}} \rightarrow \mathcal{M}_B$ , and we call it the *cohom functor*.  $M$  is said to be  $(B, \mathfrak{C})$ -*injector* if the functor  $-\otimes_B M : \mathcal{M}_B \rightarrow \mathcal{M}^{\mathfrak{C}}$  preserves injective objects. Of course, if  $B$  is a QF ring and  $M$  is  $(B, \mathfrak{C})$ -injector then  $M$  is injective in  $\mathcal{M}^{\mathfrak{C}}$ .



**Lemma 3.2.** *Let  $\mathfrak{C}$  be an  $A$ -coring such that  ${}_A\mathfrak{C}$  is flat. If  $0 \neq M \in {}_B\mathcal{M}^\mathfrak{C}$  is quasi-finite, such that*

- (a)  *$B$  is semisimple and  $\mathcal{M}^\mathfrak{C}$  is locally finitely generated; or*
- (b)  *$B$  is semiperfect and there is a finitely generated projective  $M_0$  in  $\mathcal{M}^\mathfrak{C}$  such that  $\text{Hom}_\mathfrak{C}(M_0, M) \neq 0$ . (the last condition is fulfilled if  $\mathfrak{C}$  is cosemisimple (=  $\mathcal{M}^\mathfrak{C}$  is a discrete spectral category, see [24, 6], or if  $A$  is right artinian,  ${}_A\mathfrak{C}$  is projective, and  $\mathfrak{C}$  is semiperfect, see [7, Theorem 3.1].);*

*then  $M$  has IBN as a  $B - \mathfrak{C}$ -bicomodule.*

*In particular, if  $A$  is semisimple, or  $A$  is semiperfect and there is a non-zero finitely generated projective in  $\mathcal{M}^\mathfrak{C}$ , then  $0 \neq \mathfrak{C}$  has IBN as a  $A - \mathfrak{C}$ -bicomodule.*

*Proof.* We prove at the same time the two statements. Let  $M_0$  be a finitely generated subcomodule of  $M$  (resp. a finitely generated projective in  $\mathcal{M}^\mathfrak{C}$  such that  $\text{Hom}_\mathfrak{C}(M_0, M) \neq 0$ ). We have  $\text{h}_\mathfrak{C}(M, M_0)^* = \text{Hom}_B(\text{h}_\mathfrak{C}(M, M_0), B) \simeq \text{Hom}_\mathfrak{C}(M_0, M)$  as left  $B$ -modules. Since the functor  $- \otimes_B M : \mathcal{M}_B \rightarrow \mathcal{M}^\mathfrak{C}$  is exact and preserves coproducts, the cohom functor  $\text{h}_\mathfrak{C}(\Lambda, -) : \mathcal{M}^\mathfrak{C} \rightarrow \mathcal{M}_B$  preserves finitely generated (resp. finitely generated projective) objects. In particular,  $\text{h}_\mathfrak{C}(M, M_0)$  is a finitely generated (resp. finitely generated projective) right  $B$ -module. Hence the left  $B$ -module  $\text{Hom}_\mathfrak{C}(M_0, M)$  is so.

Now suppose that  $M^m \simeq M^n$ . Since the functor  $\text{Hom}_\mathfrak{C}(M_0, -) : {}_B\mathcal{M}^\mathfrak{C} \rightarrow {}_B\mathcal{M}$  is  $k$ -linear, then  $\text{Hom}_\mathfrak{C}(M_0, M^m) \simeq \text{Hom}_\mathfrak{C}(M_0, M^n)$  as left  $B$ -bimodules. Hence  $\text{Hom}_\mathfrak{C}(M_0, M)^m \simeq \text{Hom}_\mathfrak{C}(M_0, M)^n$  as left  $B$ -bimodules. Finally, by [1, Theorem 27.11],  $m = n$ .

For the particular case we take  $M = \mathfrak{C}$ . □

**Corollary 3.3.** *Let  $C$  be a  $k$ -coalgebra such that  ${}_kC$  is flat. If  $0 \neq M \in \mathcal{M}^C$  is quasi-finite, such that*

- (a)  *$k$  is semisimple; or*
- (b)  *$k$  is semiperfect and there is a finitely generated projective  $M_0$  in  $\mathcal{M}^C$  such that  $\text{Hom}_C(M_0, M) \neq 0$ ;*

*then  $M$  has IBN as a  $C$ -comodule.*

*In particular, if  $k$  is semisimple, or  $k$  is semiperfect and there is a non-zero finitely generated projective in  $\mathcal{M}^C$ , then  $0 \neq C$  has IBN as an  $C$ -bicomodule.*

**Proposition 3.4.** *Let  $\mathfrak{C} \neq 0$  be an  $A$ -coring such that  $A$  is semisimple, or  $A$  is semiperfect and there is a non-zero finitely generated projective in  $\mathcal{M}^\mathfrak{C}$ . If every  $\mathfrak{C}$ -bicomodule which is  $(A, \mathfrak{C})$ -injector, is isomorphic to  $\mathfrak{C}^{(I)}$  as  $A - \mathfrak{C}$ -bicomodules for some set  $I$ , then the coring  $\mathfrak{C}$  has right Aut-Pic.*

*Proof.* The proof is analogous to that of [10, Proposition 2.4] and the proof of Bolla ([4, p. 264]) of the fact that every ring such that all left progenerators are free has Aut-Pic.

Let  $M$  be a right invertible  $\mathfrak{C}$ -bicomodule. By assumptions,  $M \simeq \mathfrak{C}^{(I)}$  as  $A - \mathfrak{C}$ -bicomodules for some set  $I$ . Let  $M_0$  be a finitely generated (resp. finitely generated

projective) right comodule. Since  $M_0$  is a small object (see [20]), then the  $k$ -linear functor  $\text{Hom}_{\mathfrak{C}}(M_0, -) : \mathcal{M}^{\mathfrak{C}} \rightarrow \mathcal{M}_k$  preserves coproducts. Therefore,  $\text{Hom}_{\mathfrak{C}}(M_0, M) \simeq \text{Hom}_{\mathfrak{C}}(M_0, \mathfrak{C})^{(I)}$  as left  $A$ -modules. Hence  $I$  is a finite set, and  $M \simeq \mathfrak{C}^{(n)}$  as  $A - \mathfrak{C}$ -bicomodules for some  $n \geq 1$ . Let  $N$  be a bicomodule such that  $(N)$  is the inverse of  $(M)$  in  $\text{Pic}_k^r(\mathfrak{C})$ , then

$$\mathfrak{C} \simeq M \square_{\mathfrak{C}} N \simeq \mathfrak{C}^{(n)} \square_{\mathfrak{C}} N \simeq N^{(n)}$$

as  $A - \mathfrak{C}$ -bicomodules. On the other hand,  $N \simeq \mathfrak{C}^{(m)}$  as  $A - \mathfrak{C}$ -bicomodules for some set  $m \geq 1$ . It follows that  $\mathfrak{C} \simeq \mathfrak{C}^{(nm)}$  as  $A - \mathfrak{C}$ -bicomodules. By Lemma 3.2,  $nm = 1$  and then  $M \simeq \mathfrak{C}$  as  $A - \mathfrak{C}$ -bicomodules. Finally, from Proposition 2.7,  $M \simeq {}_f \mathfrak{C}$  as bicomodules for some  $f = (\varphi, 1_A) \in \text{Aut}_k(\mathfrak{C})$ .  $\square$

**Corollary 3.5.** *Let  $C$  be a  $k$ -coalgebra such that  $k$  is semisimple, or  $k$  is a QF ring and there is a non-zero finitely generated projective in  $\mathcal{M}^C$ . If every right injective  $C$ -comodule is free, then the coalgebra  $C$  has right Aut-Pic.*

The following result allow us to simplify the computation of the Picard group of some interesting corings.

**Proposition 3.6.** (1) *Let  ${}_B \Sigma_A$  be a bimodule such that  $\Sigma_A$  is finitely generated and projective and  ${}_B \Sigma$  is faithfully flat. Then  $\text{Pic}_k^r(\Sigma^* \otimes_B \Sigma) \simeq \text{Pic}_k(B)$ , where  $\Sigma^* \otimes_B \Sigma$  is the comatrix coring [12] associated to  $\Sigma$ . If moreover  $B$  has Aut-Pic then  $\text{Pic}_k^r(\Sigma^* \otimes_B \Sigma) \simeq \text{Out}_k(B)$ .*

*In particular, For a  $k$ -algebra  $A$  and  $n \in \mathbb{N}$ , we have  $\text{Pic}_k^r(M_n^c(A)) \simeq \text{Pic}_k(A)$ , where  $M_n^c(A)$  is the  $(n, n)$ -matrix coring over  $A$  defined in [6, 17.7]. If moreover  $A$  has Aut-Pic then  $\text{Pic}_k^r(M_n^c(A)) \simeq \text{Out}_k(A)$ .*

(2) *Let  $\mathfrak{C}$  be an  $A$ -coring such that  $\mathfrak{C}_A$  is flat, and  $R$  the opposite algebra of  ${}^* \mathfrak{C}$ . If  ${}_A \mathfrak{C}$  is finitely generated projective (e.g. if  $\mathfrak{C}$  is a Frobenius ring, see [6]), then  $\text{Pic}_k^r(\mathfrak{C}) \simeq \text{Pic}_k(R)$ .*

(3) *If  $\mathfrak{C}$  is an  $A$ -coring such that the category  $\mathcal{M}^{\mathfrak{C}}$  has a finitely generated projective generator  $U$ , then  $\text{Pic}_k^r(\mathfrak{C}) \simeq \text{Pic}_k(\text{End}_{\mathfrak{C}}(U))$ .*

*Proof.* (1) From [12, Theorem 3.10], we have an equivalence of categories  $- \otimes_B \Sigma : \mathcal{M}_B \rightarrow \mathcal{M}^{\Sigma^* \otimes_B \Sigma}$ . Theorem 2.5 achieves then the proof. Now we prove the particular case. Let  $A$  be a  $k$ -algebra and  $n \in \mathbb{N}$ . If we take  $\Sigma = A^n$ , the comatrix coring  $\Sigma^* \otimes_B \Sigma$  can be identified with  $M_n^c(A)$ .

(2) From [5, Lemma 4.3], the categories  $\mathcal{M}^{\mathfrak{C}}$  and  $\mathcal{M}_R$  are isomorphic to each other. It is enough to apply Theorem 2.5.

(3) This follows immediately from an alternative of Gabriel-Popescu's Theorem (see [24, p. 223]) and Theorem 2.5.  $\square$

## 4 Application to the Picard group of $gr - (A, X, G)$

In this Section we adopt the notations of [21] and [8].

We recall from [6], that a *right-right entwining structure over  $k$*  is a triple  $(A, C, \psi)$ , where  $A$  is a  $k$ -algebra,  $C$  is a  $k$ -coalgebra, and  $\psi : C \otimes A \rightarrow A \otimes C$  is a  $k$ -linear map, such that

$$(ES1) \quad \psi \circ (1_C \otimes m) = (m \otimes 1_C) \circ (1_A \otimes \psi) \circ (\psi \otimes 1_A), \text{ or equivalently, for all } a, b \in A, c \in C, \\ \sum (ab)_\psi \otimes c^\psi = \sum a_\psi b_\Psi \otimes c^{\psi\Psi},$$

$$(ES2) \quad (1_a \otimes \Delta) \circ \psi = (\psi \otimes 1_C) \circ (1_C \otimes \psi) \circ (\Delta \otimes 1_A), \text{ or equivalently, for all } a \in A, c \in C, \\ \sum a_\psi \otimes \Delta(c^\psi) = \sum a_{\psi\Psi} \otimes c_{(1)}^\Psi \otimes c_{(2)}^\psi,$$

$$(ES3) \quad \psi \circ (1_C \otimes \eta) = \eta \otimes 1_C, \text{ or equivalently, for all } c \in C, \sum (1_A)_\psi \otimes c^\psi = 1_A \otimes c,$$

$$(ES4) \quad (1_A \otimes \epsilon) \circ \psi = \epsilon \otimes 1_A, \text{ or equivalently, for all } a \in A, c \in C, \sum a_\psi \epsilon(c^\psi) = a \epsilon(c).$$

where  $m$  and  $\eta$  are respectively the multiplication and the unit maps of  $A$ , and  $\psi(c \otimes a) = a_\psi \otimes c^\psi = a_\Psi \otimes c^\Psi$ .

A *morphism*  $(A, C, \psi) \rightarrow (A', C', \psi')$  is a pair  $(\alpha, \gamma)$  with  $\alpha : A \rightarrow A'$  is a morphism of algebras, and  $\gamma : C \rightarrow C'$  is a morphism of coalgebras such that  $(\alpha \otimes \gamma) \circ \psi = \psi' \circ (\gamma \otimes \alpha)$  or equivalently, for all  $a \in A, c \in C, \sum \alpha(a_\psi) \otimes \gamma(c^\psi) = \sum \alpha(a)_{\psi'} \otimes \gamma(c)^{\psi'}$ . We denote this category by  $\mathbb{E}_\bullet(k)$ . Notice that  $(\alpha, \gamma) \in \mathbb{E}_\bullet(k)$  is an isomorphism if and only if  $\alpha$  and  $\gamma$  are bijective.

Again we recall from [6], that a *right-right entwined modules* over a right-right entwining structure  $(A, C, \psi)$  is a  $k$ -module  $M$  which is a right  $A$ -module with multiplication  $\psi_M$ , and a right  $C$ -comodule with comultiplication  $\rho_M$  such that  $(\psi_M \otimes 1_C) \circ (1_M \otimes \psi) \circ (\rho_M \otimes 1_A) = \rho_M \circ \psi_M$ , or equivalently, for all  $m \in M$  and  $a \in A, \rho_M(ma) = \sum m_{(0)} a_\psi \otimes m_{(1)}^\psi$ . A *morphism* between entwined modules is a morphism of right  $A$ -modules and right  $C$ -comodules at the same time. We denote this category by  $\mathcal{M}(\psi)_A^C$ .

The following result due to Takeuchi shows that entwining structures and entwined modules are very related to corings and comodules over corings respectively. For a better understanding, we add a sketch of the proof.

**Theorem 4.1.** (a) Let  $A$  be an algebra,  $C$  be a  $k$ -coalgebra, and let  $\psi : C \otimes A \rightarrow A \otimes C$  be a  $k$ -linear map. Obviously  $A \otimes C$  has a structure of left  $A$ -module by  $b(a \otimes c) = ba \otimes c$  for  $a, b \in A$  and  $c \in C$ . Define the right  $A$ -module action on  $A \otimes C$ ,  $\psi_{A \otimes C}^r : A \otimes C \otimes A \xrightarrow{A \otimes \psi} A \otimes A \otimes C \xrightarrow{m \otimes C} A \otimes C$ , that is,  $(a \otimes c)b = a\psi(c \otimes b)$  for  $a, b \in A$  and  $c \in C$ . Define also

$$\Delta : A \otimes C \xrightarrow{A \otimes \Delta_C} A \otimes C \otimes C \simeq (A \otimes C) \otimes_A (A \otimes C)$$

(for every  $a \in A, c \in C, \Delta(a \otimes c) = \sum (a \otimes c_{(1)}) \otimes_A (1_A \otimes c_{(2)})$ , where  $\Delta_C(c) = \sum c_{(1)} \otimes c_{(2)}$ , and

$$\epsilon : A \otimes C \xrightarrow{A \otimes \epsilon_C} A \otimes k \simeq A$$

$(\epsilon(a \otimes c) = a\epsilon_C(c))$ . Thus,  $(A \otimes C, \Delta, \epsilon)$  is an  $A$ -coring if and only if  $(A, C, \psi)$  is an entwining structure.

(b) Let  $(A, C, \psi)$  be an entwining structure,  $M$  be a right  $A$ -module with the action  $\psi_M : M \otimes A \rightarrow M$ , and let  $\rho_M : M \rightarrow M \otimes C$  be a  $k$ -linear map. Define

$$\rho'_M : M \xrightarrow{\rho_M} M \otimes C \simeq M \otimes_A (A \otimes C)$$

(for every  $m \in M$ ,  $\rho'_M(m) = \sum m_{(0)} \otimes_A (1_A \otimes m_{(1)})$ , where  $\rho_M(m) = \sum m_{(0)} \otimes m_{(1)}$ ). Then,  $(M, \rho'_M)$  is a right  $A \otimes C$ -comodule if and only if  $(M, \psi_M, \rho_M)$  is a right-right entwined module over  $(A, C, \psi)$ . Hence, the category of right  $A \otimes C$ -comodules,  $\mathcal{M}^{A \otimes C}$  is isomorphic to the category of right-right entwined modules over  $(A, C, \psi)$ ,  $\mathcal{M}(\psi)_A^C$ .

*Proof.* (a) It is easy to verify that

- For all  $a, b \in A, c \in C$ ,  $(1 \otimes c)(ab) = ((1 \otimes c)a)b \iff (\text{ES1})$ .
- For all  $c \in C$ ,  $(1 \otimes c)1 = 1 \otimes c \iff (\text{ES3})$ .
- For all  $a \in A, c \in C$ ,  $\Delta((1 \otimes c)a) = \Delta(1 \otimes c)a \iff (\text{ES2})$ .
- For all  $a \in A, c \in C$ ,  $\epsilon((1 \otimes c)a) = \epsilon(1 \otimes c)a \iff (\text{ES4})$ .

Moreover, the coassociativity of  $\Delta$  and the counit property of  $\epsilon$  follow from that of  $\Delta_C$  and  $\epsilon_C$ . Hence (a) follows.

(b) It is easy to verify that

- $\rho'_M$  is  $A$ -linear if and only if for all  $m \in M$  and  $a \in A$ ,  $\rho_M(ma) = \sum m_{(0)} a_\psi \otimes m_{(1)}^\psi$ .
- $\rho'_M$  is coassociative if and only if  $\rho_M$  is coassociative.
- The counit property of  $\epsilon$  holds if and only if that of  $\epsilon_C$  holds. Hence (b) follows. □

It is easy to verify that if  $(\alpha, \gamma) : (A, C, \psi) \rightarrow (A', C', \psi')$  is a morphism of entwined structures, then  $(\alpha \otimes \gamma, \alpha) : A \otimes C \rightarrow A' \otimes C'$  is a morphism of corings. Hence we have a functor

$$F : \mathbb{E}_\bullet(k) \rightarrow \mathbf{Crg}_k.$$

By Theorem 2.5, we obtain

**Proposition 4.2.** *For an entwining structure  $(A, C, \psi)$  there is an exact sequence*

$$1 \longrightarrow \text{Ker}(\Omega \circ F) \longrightarrow \text{Aut}_k((A, C, \psi)) \xrightarrow{\Omega \circ F} \text{Pic}_k^r(A \otimes C) \simeq \text{Pic}_k^r(\mathcal{M}(\psi)_A^C),$$

and  $\text{Ker}(\Omega \circ F)$  is the set of  $(\alpha, \gamma) \in \text{Aut}_k((A, C, \psi))$  such that there is  $p \in (A \otimes C)^*$  satisfying  $\sum (\alpha(a) \otimes \gamma(c_{(1)}))p(1_A \otimes c_{(2)}) = \sum p(a \otimes c_{(1)}) \otimes c_{(2)}$  for all  $a \in A, c \in C$ .

A right-right *Doi-Koppinen structure* or simply *DK structure* over  $k$  [8] is a triple  $(H, A, C)$ , where  $H$  is a bialgebra,  $A$  is a right  $H$ -comodule algebra, and  $C$  is a right  $H$ -module coalgebra. A *morphism of DK structures* is a triple  $(\hbar, \alpha, \gamma) : (H, A, C) \rightarrow (H', A', C')$ , where  $\hbar : H \rightarrow H'$ ,  $\alpha : A \rightarrow A'$ , and  $\gamma : C \rightarrow C'$  are respectively a bialgebra morphism, an algebra morphism, and a coalgebra morphism such that  $\rho_{A'}(\alpha(a)) = \alpha(a_{(0)}) \otimes \hbar(a_{(1)})$  and  $\gamma(ch) = \gamma(c)\hbar(h)$ , for all  $a \in A, c \in C, h \in H$ . This yields a category which we denote by  $\mathbb{DK}_{\bullet}(k)$ . Moreover, we have a faithful functor  $G : \mathbb{DK}_{\bullet}(k) \rightarrow \mathbb{E}_{\bullet}(k)$  defined by  $G((H, A, C)) = (A, C, \psi)$  with  $\psi : C \otimes A \rightarrow A \otimes C$  and  $\psi(c \otimes a) = a_{(0)} \otimes ca_{(1)}$ , and  $G((\hbar, \alpha, \gamma)) = (\alpha, \gamma)$ . (see [8, Proposition 17].) Notice that  $(\hbar, \alpha, \gamma) \in \mathbb{DK}_{\bullet}(k)$  is an isomorphism if and only if  $\hbar, \alpha$  and  $\gamma$  are bijective. The category of right *Doi-Koppinen-Hopf modules* over the right-right DK structure  $(H, A, C)$  is exactly the category of  $\mathcal{M}(\psi)_A^C$ , and it is denoted by  $\mathcal{M}(H)_A^C$ .

Hence Proposition 4.2 yields the following

**Proposition 4.3.** *For a DK structure  $(H, A, C)$  there is an exact sequence*

$$1 \longrightarrow \text{Ker}(\Omega \circ F \circ G) \longrightarrow \text{Aut}_k((H, A, C)) \xrightarrow{\Omega \circ F \circ G} \text{Pic}_k^r(A \otimes C) \simeq \text{Pic}_k^r(\mathcal{M}(H)_A^C),$$

and  $\text{Ker}(\Omega \circ F \circ G)$  is the set of  $(\hbar, \alpha, \gamma) \in \text{Aut}_k((H, A, C))$  such that there is  $p \in (A \otimes C)^*$  satisfying  $\sum (\alpha(a) \otimes \gamma(c_{(1)}))p(1_A \otimes c_{(2)}) = \sum p(a \otimes c_{(1)}) \otimes c_{(2)}$  for all  $a \in A, c \in C$ .

Finally, we consider the category of right modules graded by a  $G$ -set,  $gr - (A, X, G)$ , where  $G$  is a group and  $X$  is a right  $G$ -set. This category is introduced and studied in [17]. A study of the graded ring theory can be found in the recent book [19]. This category is equivalent to a category of modules over a ring if  $X = G$  and  $A$  is strongly graded (by Dade Theorem [19, Theorem 3.1.1]), or if  $X$  is a finite set (by [17, Theorem 2.13]). But there is an example of a category of graded modules which is not equivalent to a category of modules, see [16, Remark 2.4].

Let  $G$  be a group,  $X$  a right  $G$ -set, and  $A$  be a  $G$ -graded  $k$ -algebra. We know that  $(kG, A, C)$  is a DK structure, with  $kG$  is a Hopf algebra. Then we have,  $(A, kX, \psi)$  is an entwining structure where  $\psi : kX \otimes A \rightarrow A \otimes kX$  is the map defined by  $\psi(x \otimes a_g) = a_g \otimes xg$  for all  $x \in X, g \in G, a_g \in A_g$ . From the above considerations, we have an  $A$ -coring,  $A \otimes kX$ . The comultiplication and the counit maps of the coring  $A \otimes kX$  are defined by:

$$\Delta_{A \otimes kX}(a \otimes x) = (a \otimes x) \otimes_A (1_A \otimes x), \quad \epsilon_{A \otimes kX}(a \otimes x) = a \quad (a \in A, x \in X).$$

We know that this coring is coseparable (see [26, Section 6]). From [8, §4.6],  $gr - (A, X, G) \simeq \mathcal{M}(kG)_A^{kX}$ . Moreover, the left  $A$ -module  $A \otimes kX$  is free with the basis  $\{1_A \otimes x \mid x \in X\}$ . It is easy to see that the last family is a basis of the right  $A$ -module  $A \otimes kX$  (since  $a_g \otimes x = (1_A \otimes xg^{-1})a_g$  for every  $g \in G, a_g \in A_g, x \in X$ , and since  $\psi$  is an isomorphism.) Each  $p \in (A \otimes kX)^*$  is entirely determined by the data of  $p(1_A \otimes x)$ , for all  $x \in X$ . The same thing holds for  $\varphi$ , where  $(\varphi, \rho) \in \text{Aut}_k(A \otimes kX)$ .

By Theorem 2.5 we get

**Proposition 4.4.** (a) We have,  $p \in (A \otimes kX)^*$  is invertible if and only if there exists  $q \in (A \otimes kX)^*$  such that  $\sum_{h \in G} q(1_A \otimes xh^{-1})p(1_A \otimes x)_h = 1_A = \sum_{h \in G} p(1_A \otimes xh^{-1})q(1_A \otimes x)_h$  for every  $x \in X$ .

(b) We have an exact sequence

$$1 \longrightarrow \text{Ker}(\Omega) \longrightarrow \text{Aut}_k(A \otimes kX) \xrightarrow{\Omega} \text{Pic}_k(A \otimes kX) \simeq \text{Pic}_k(\text{gr} - (A, X, G)),$$

and  $(\varphi, \rho) \in \text{Ker}(\Omega)$  if and only if there exists  $p \in (A \otimes kX)^*$  invertible such that

- (i) for all  $x \in X$ ,  $a_y^x p(1_A \otimes x)_h = 0$  for all  $y \in X, h \in G$  such that  $yh \neq x$ , where  $\varphi(1_A \otimes x) = \sum_{y \in X} a_y^x \otimes y$ , and
- (ii) for all  $x \in X, a \in A$ ,  $p(a \otimes x) = \rho(a)p(1_A \otimes x)$ .

*Proof.* We have,  $p \in (A \otimes kX)^*$  is invertible if and only if there exists  $q \in (A \otimes kX)^*$  such that

$$q(p(a \otimes x)(1_A \otimes x)) = \epsilon(a \otimes x) = a \quad (2)$$

$$p(q(a \otimes x)(1_A \otimes x)) = \epsilon(a \otimes x) = a, \quad (3)$$

for all  $a \in A, x \in X$ . (from [6, 17.8].) On the other hand,

$$\begin{aligned} q(p(a_g \otimes x)(1_A \otimes x)) &= q(p(1_A \otimes xg^{-1})(a_g \otimes x)) \\ &= q\left(\sum_{h \in G} (b_h a_g) \otimes x\right), \quad \text{where } p(1_A \otimes xg^{-1}) = \sum_{h \in G} b_h \\ &= \sum_{h \in G} q(1_A \otimes x(hg)^{-1})(b_h a_g), \quad \text{since } b_h a_g \in A_{hg} \\ &= \sum_{h \in G} q(1_A \otimes xg^{-1}h^{-1})p(1_A \otimes xg^{-1})_h a_g, \end{aligned}$$

for all  $x \in X, g \in G, a_g \in A_g$ . Hence

$$\begin{aligned} (2) &\iff q(p(a_g \otimes x)(1_A \otimes x)) = a_g \quad \text{for all } x \in X, g \in G, a_g \in A_g \\ &\iff \sum_{h \in G} q(1_A \otimes xh^{-1})p(1_A \otimes x)_h = 1_A \quad \text{for all } x \in X. \end{aligned}$$

By symmetry,

$$(3) \iff \sum_{h \in G} p(1_A \otimes xh^{-1})q(1_A \otimes x)_h = 1_A \quad \text{for all } x \in X.$$

By Theorem 2.5,  $(\varphi, \rho) \in \text{Ker}(\Omega)$  if and only if there exists  $p \in (A \otimes kX)^*$  invertible with  $\varphi(a \otimes x)p(1_A \otimes x) = p(a \otimes x) \otimes x$  for all  $a \in A, x \in X$ . On the other hand,

$$\begin{aligned} \varphi(a \otimes x)p(1_A \otimes x) &= \rho(a)\varphi(1_A \otimes x)p(1_A \otimes x) \\ &= \rho(a) \sum_{y \in X} \sum_{h \in G} (a_y^x \otimes y) b_h, \quad \text{where } \varphi(1_A \otimes x) = \sum_{y \in X} a_y^x \otimes y, p(1_A \otimes x) = \sum_{h \in G} b_h \\ &= \rho(a) \sum_{y \in X} \sum_{h \in G} a_y^x b_h \otimes yh. \end{aligned}$$

Since  $\sum_{y \in X} a_y^x = \epsilon \circ \varphi(1_A \otimes x) = \rho \circ \epsilon(1_A \otimes x) = 1_A$ , (b) follows.  $\square$

Now, let  $f : G \rightarrow G'$  be a morphism of groups,  $X$  a right  $G$ -set,  $X'$  a right  $G'$ -set, and  $\varphi : X \rightarrow X'$  a map. Let  $A$  be a  $G$ -graded  $k$ -algebra,  $A'$  a  $G'$ -graded  $k$ -algebra, and  $\alpha : A \rightarrow A'$  a morphism of algebras. We have,  $\hbar : kG \rightarrow kG'$  defined by  $\hbar(g) = f(g)$  for each  $g \in G$ , is a morphism of Hopf algebras, and  $\gamma : kX \rightarrow kX'$  defined by  $\gamma(x) = \varphi(x)$  for each  $x \in X$ , is a morphism of coalgebras. It is easy to show that  $(\hbar, \alpha, \gamma)$  is a morphism of DK structures if and only if

$$\varphi(xg) = \varphi(x)f(g) \quad \text{for all } g \in G, x \in X \quad (4)$$

$$\alpha(A_g) \subset A'_{f(g)} \quad \text{for all } g \in G. \quad (5)$$

We define a category as follows. The objects are the triples  $(G, X, A)$ , where  $G$  is a group,  $X$  is a right  $G$ -set, and  $A$  is a graded  $k$ -algebra. The morphisms are the triples  $(f, \varphi, \alpha)$ , where  $f : G \rightarrow G'$  is a morphism of groups,  $\varphi : X \rightarrow X'$  is a map, and  $\alpha : A \rightarrow A'$  is a morphism of algebras such that (6), (7) hold. We denote this category by  $\mathbb{G}^r(k)$ . There is a faithful functor

$$H : \mathbb{G}^r(k) \rightarrow \mathbb{DK}_\bullet(k)$$

defined by  $H((G, X, A)) = (kG, A, kX)$  and  $H((f, \varphi, \alpha)) = (\hbar, \alpha, \gamma)$ . We obtain then the next result. We think that our subgroup of  $\text{Pic}_k(\text{gr} - (A, X, G))$  is more simple than the Beattie-Del Río subgroup (see [3, §2]).

**Proposition 4.5.** *There is an exact sequence*

$$1 \longrightarrow \text{Ker}(\Omega \circ F \circ G \circ H) \longrightarrow \text{Aut}_k((G, X, A)) \xrightarrow{\Omega \circ F \circ G \circ H} \text{Pic}_k(A \otimes kX) \simeq \text{Pic}_k(\text{gr} - (A, X, G)),$$

and  $\text{Ker}(\Omega \circ F \circ G \circ H)$  is the set of  $(f, \varphi, \alpha) \in \text{Aut}_k((G, X, A))$  such that there is  $p \in (A \otimes kX)^*$  satisfying  $p(a \otimes x) = \alpha(a)p(1_A \otimes x)$  for all  $x \in X, a \in A$ ; and  $p(1_A \otimes x)_h = 0$  for all  $x \in X, h \in G$  such that  $\varphi(x)h \neq x$ .

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## References

- [1] F. W. Anderson and K. R. Fuller, *Rings and categories of modules*, Graduate texts in Math. 13, Springer-Verlag, Second edition, 1992.
- [2] H. Bass, *Algebraic K-Theory*, W. A. Benjamin, Inc., 1968.
- [3] M. Beattie and A. Del Río, *The Picard group of a category of graded modules*, Comm. Algebra **24**(14) (1996), 4397–4414.

- [4] M. L. Bolla, *Isomorphisms between endomorphism rings of progenerators*, J. Algebra **87** (1984), 261–281.
- [5] T. Brzeziński, *The structure of corings: Inductions functors, Maschke type theorem, and Frobenius and Galois-type properties*. Algebras and Repr. Theory, **5** (2002), 389–410.
- [6] T. Brzeziński and R. Wisbauer, *Corings and comodules*, Cambridge University Press, Cambridge, 2003.
- [7] S. Caenepeel and M. Iovanov, *Comodules over semiperfect corings*, in "Proceedings of the International Conference on Mathematics and its Applications, ICMA 2004", S.L. Kalla and M.M. Chawla (Eds.), Kuwait University, Kuwait, 2005, 135–160.
- [8] S. Caenepeel, G. Militaru, and S. Zhu, *Frobenius and separable functors for generalized module categories and nonlinear equations*, Lec. Not. Math. 1787, Springer Verlag, Berlin (2002).
- [9] W. Chin and S. Montgomery, *Basic coalgebras*, Collection Modular interfaces, Riverside, CA, 1995, 41–47.
- [10] J. Cuadra, J. R. García Rozas and B. Torrecillas, *Outer automorphisms and Picard groups of coalgebras*, Rev. Roumaine Math. Pures Appl. **45** (2000), 775–788.
- [11] C. W. Curtis and I. Reiner, *Methods of representation theory with applications to finite groups and orders*, volume II, John Wiley & Sons, New York, 1987.
- [12] L. El Kaoutit and J. Gómez-Torrecillas, *Comatrix corings: Galois corings, descent theory, and a structure theorem for cosemisimple corings*, Math. Z. **244** (2003), 887–906.
- [13] A. Fröhlich, *The Picard group of noncommutative rings, in particular of orders*, Trans. Amer. Math. Soc. **180** (1973), 1–45.
- [14] J. Gómez-Torrecillas, *Separable functors in corings*, Inter. J. Math. Math. Sci. **30** (2002), 203–225.
- [15] T. Y. Lam, *Lectures on modules and rings*, Graduate texts in Math. 189, Springer-Verlag, 1999.
- [16] C. Menini and C. Năstăsescu, *When is  $R\text{-gr}$  equivalent to the category of modules?*, J. Pure Appl. Algebra, **51**, (1988), 277–291.
- [17] C. Năstăsescu, S. Raianu, and F. Van Oystaeyen, *Graded modules over  $G$ -Sets*. Math. Z. **203** (1990), 605–627.
- [18] C. Năstăsescu, B. Torrecillas and F. Van Oystaeyen, *IBN for graded rings*, Comm. Algebra **8** (2000), 1351–1360.



- [19] C. Năstăsescu and F. Van Oystaeyen, *Methods of graded rings*. Lecture Notes Math. 1836, Springer Verlag, Berlin (2004).
- [20] N. Popescu, *Abelian categories with applications to rings and modules*, Academic Press, London, 1973.
- [21] A. Del Río, *Categorical methods in graded ring theory*. Publ. Math. **36** (1992), 489–531.
- [22] A. Rosenberg and D. Zelinsky, *Automorphisms of separable algebras*, Pacific J. math., **11** (1961), 1109–1117.
- [23] B. Torrecillas and Y. H. Zhang, *The Picard groups of coalgebras*, Comm. Algebra **24**(7) (1996), 2235–2247.
- [24] B. Stenström, *Rings of quotients*, Springer-verlag, Berlin, 1975.
- [25] M. Sweedler, *The predual theorem to the Jacobson-Bourbaki theorem*, Trans. Amer. Math. Soc. **213** (1975), 391–406.
- [26] M. Zarouali-Darkaoui, *Adjoint and Frobenius pairs of functors for corings*, Comm. Algebra, to appear, see arXiv:math.RA/0505478.
- [27] M. Zarouali-Darkaoui, *Comatrix coring generalized and equivalences of categories of comodules*, Preprint 2005, arXiv:math.RA/0505534.

Department of Algebra, University of Granada,  
 E18071 Granada, Spain  
 E-mail address: zaroual@correo.ugr.es